Understand the real numbers

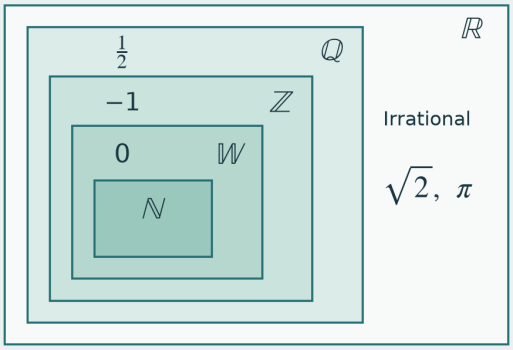
LESSON 1.1 / Real numbers and the number line

Number sets fit inside one another

$$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

- Natural numbers here begin at 1; whole numbers also include 0.
- Integers add the negative whole numbers.
- An integer is also rational and real.

Nested sets inside the real numbers



Natural numbers here begin at 1

Numbers can belong to several sets at once. In this course, natural numbers begin at one. Whole numbers include zero as well. Integers also include negative whole numbers. Every integer is rational because it can be written over one, and every rational number is real. The sets are nested. When a question asks for the most specific set, give the smallest named set containing the value, rather than stopping at real.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Classify the value, not its appearance

$$-12 = \frac{-12}{1}, \qquad \sqrt{81} = 9$$

- Simplify an expression before classifying it.
- A negative integer is rational, even without a fraction bar.
- A square-root symbol does not automatically mean irrational.

Simplify first, then classify

-12

Integer → rational → real

$0.375 = \frac{3}{8}$

Rational → real

$\sqrt{81} = 9$

Natural → whole → integer → rational → real

The appearance of a number can be misleading. Negative twelve is an integer, but it is also rational because negative twelve divided by one is a fraction of integers. The square root of eighty-one equals nine, so its most specific set here is natural. Zero is whole but not natural under our stated convention. Simplify first, then follow the nested sets outward if all memberships are requested.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Rational means a ratio of integers

$$\frac{p}{q}, \quad p, q \in \mathbb{Z}, \quad q \neq 0$$

- A terminating decimal is rational.
- A repeating decimal is rational, even if it never ends.
- Irrational decimals neither terminate nor eventually repeat.

Ending, repeating or neither?

$0.375 = \frac{3}{8}$

Terminating: rational

$0.\overline{3} = \frac{1}{3}$

Repeating: rational

$\sqrt{2} = 1.41421356\ldots$

Neither ends nor repeats: irrational

A rational number can be written as a ratio of two integers with a nonzero denominator. A terminating decimal such as zero point three seven five equals three eighths. A repeating decimal such as zero point three repeating equals one third. It never ends, but it is rational. Irrational numbers have decimal expansions that neither end nor settle into a repeating block. Never use goes on forever as the only test.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Exact value and approximation are different

$$\sqrt{2} \approx 1.414, \qquad \sqrt{2} \neq 1.414$$

- √2 and π are exact irrational numbers.
- 1.414 and 3.14 are terminating rational decimals.
- Use ≈ for an approximation, not =.

The approximation is a different value

$\sqrt{2}$ irrational

$\approx$

$1.414 = \frac{707}{500}$

The written decimal is rational

Close to √2 does not mean equal to √2

The square root of two is irrational, but one point four one four is rational: it is an exact terminating decimal. That decimal approximates the square root; it is not equal to it. The same distinction applies to pi and three point one four. A calculator display usually shows a rounded approximation, not the entire decimal expansion. Classify the value actually written and use the approximation sign when the two values are only close.

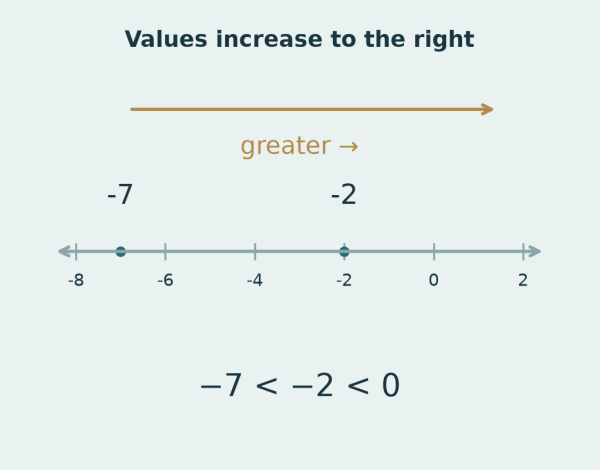
Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Greater means farther right

$$-7 < -2 < 0$$

- Values increase as you move right.
- A negative value closer to zero is greater.
- Distance from zero and numerical order answer different questions.



On a number line, values increase to the right. Negative seven lies left of negative two, so negative seven is smaller. Even though seven is larger than two as a positive magnitude, the negative signs reverse that comparison. Ask which point is farther right, not which number has the larger absolute value. Use a labelled origin and equally spaced intervals so the picture supports your explanation.

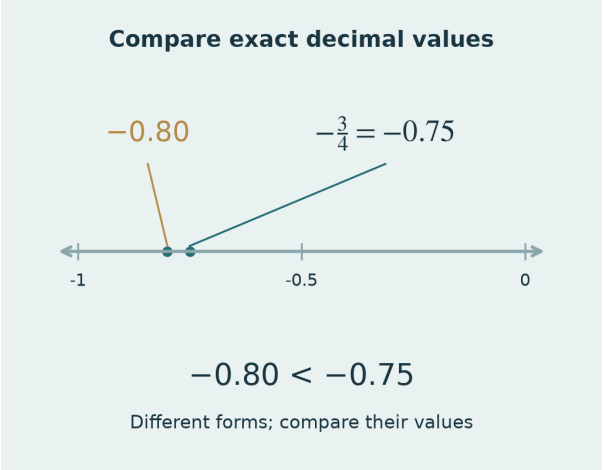
Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Give fractions and decimals a common form

$$-\frac{3}{4} = -0.75 < -0.60 = -0.6$$

- Use an exact decimal or a common denominator.
- Keep negative signs attached to the values.
- Do not round away the difference you are comparing.



To compare negative three quarters and negative zero point six, write the fraction as negative zero point seven five. The value negative zero point seven five is farther left, so negative three quarters is smaller. Converting both values into a common form makes the comparison visible. Equivalent decimals may have different written lengths, such as negative zero point six and negative zero point six zero, without changing the value.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Order a mixed list without guessing

$$-1.35 < -\frac{5}{4} < -1.2$$

- Convert -5/4 exactly to -1.25.
- Compare -1.35, -1.25 and -1.20.
- Return to the original forms in your final list.



Order negative five quarters, negative one point two and negative one point three five from least to greatest. The fraction equals negative one point two five exactly. Comparing the decimals gives negative one point three five, then negative one point two five, then negative one point two zero. Write the answer in the original forms: negative one point three five, negative five quarters, negative one point two. Check by moving right along a scaled line.

Understand the real numbers

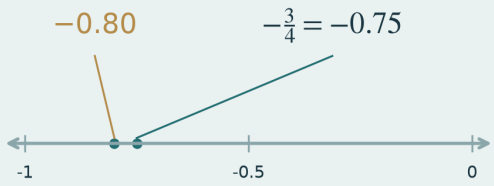
LESSON 1.1 / Real numbers and the number line

A scaled sketch can expose an error

$$-0.80 < -0.75 < 0$$

- Mark the origin and state the scale.
- Equivalent values occupy the same point.
- Explain a false claim with a numerical counterexample.

Compare exact decimal values



$-0.80 < -0.75$

Different forms; compare their values

A number-line sketch is useful only when its scale is consistent. Negative zero point eight is left of negative three quarters, because negative three quarters is negative zero point seven five. Both are left of zero. Equivalent forms, such as one half and zero point five, occupy one point rather than separate positions. A clear counterexample can disprove the claim that every negative number with a larger-looking numeral is greater.

Understand the real numbers

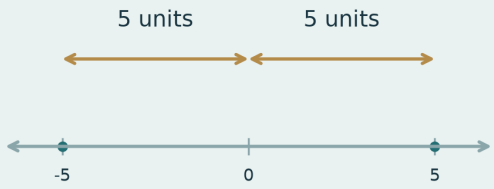
LESSON 1.1 / Real numbers and the number line

Opposites share a distance from zero

$$|-5| = |5| = 5, \qquad |0| = 0$$

- An opposite is the same distance away on the other side of 0.
- Absolute value gives that non-negative distance.
- Zero is its own opposite.

Same distance, opposite positions



$$|-5| = |5| = 5$$

$|0| = 0$; a distance cannot be negative

Opposite numbers lie the same distance from zero on opposite sides. The opposite of negative five is positive five, and both have absolute value five. Absolute value records distance, so it cannot be negative. But it can be zero: zero is zero units from itself and is its own opposite. Keep the words opposite and absolute value distinct, because they answer different questions.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Evaluate inside the bars first

$$|-8|-|3-9|=8-6=2$$

- The bars apply to the complete expression inside them.
- Work out 3-9 before taking its absolute value.
- An operation outside the bars stays outside.

Inside → absolute value → outside

$$|-8|=8$$

$$3-9=-6$$

$$|3-9|=|-6|=6$$

$$8-6=2$$

Keep the subtraction outside the bars

Evaluate the expression inside each pair of absolute-value bars before combining the results. The absolute value of negative eight is eight. Three minus nine is negative six, whose absolute value is six. The outside subtraction then gives eight minus six, which is two. Do not turn every subtraction sign into addition. For example, a negative sign outside the absolute value of negative eight would remain outside and give negative eight.

Understand the real numbers

LESSON 1.1 / Real numbers and the number line

Distance is not a signed change

$$\Delta = b - a, \qquad d = |b - a|$$

- From 5 to -3, the signed change is -8.
- The direct distance between the endpoints is 8.
- Reversing the direction changes the sign, not the distance.



A lift moves directly from floor five to floor negative three, with equally spaced floors. Its signed change is final minus initial, negative three minus five, which is negative eight floors. The distance is the absolute value, eight floor intervals. Reverse the trip and the signed change becomes positive eight, but the direct distance stays eight. If a route changes direction, total travel can be longer than the endpoint distance; state which quantity the problem asks for.

Understand the real numbers

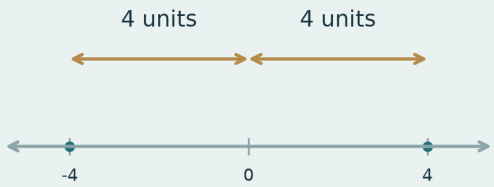
LESSON 1.1 / Real numbers and the number line

Find all points at a stated distance

$$|x| = 4 \Rightarrow x = -4 \text{ or } x = 4$$

- A positive distance from zero gives two possible positions.
- Distance zero gives only the origin.
- Explain why a negative distance gives no real position.

Two points at distance 4 from zero



$|x| = 4 : \; x = -4 \text{ or } 4$

$|0| = 0$; a distance cannot be negative

Four units from zero describes two positions, negative four and positive four. The absolute value is the same at both. If the stated distance is zero, there is just one position: zero. A negative distance describes no point. Use the number line to check that you have found every position rather than reporting only the positive one. Finish an unfamiliar problem by explaining what the symbols mean before doing arithmetic.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Classify all the sets, then the smallest

$-12 \in \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Simplify first, then classify

-12

Integer → rational → real

$0.375 = \frac{3}{8}$

Rational → real

$\sqrt{81} = 9$

Natural → whole → integer → rational → real

1. -12 is an integer; write it as -12/1 to show it is rational.

2. 0.375=375/1000=3/8 is rational but not an integer.

3. √81=9 is natural, whole, integer, rational and real.

Now you try: Classify 0 and √49 in every applicable set used in this course.

Check after attempting: 0 is whole, integer, rational and real, but not natural here. √49=7 is natural, whole, integer, rational and real.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Separate an exact value from its approximation

$\sqrt{2} \notin \mathbb{Q}, \quad 1.414 = \frac{707}{500} \in \mathbb{Q}$

The approximation is a different value

$$\sqrt{2} \quad \text{irrational}$$

$$\approx$$

$$1.414 = \frac{707}{500}$$

The written decimal is rational

Close to √2 does not mean equal to √2

1. √2 is an exact irrational value.
2. 1.414 is a terminating rational decimal.
3. Write √2≈1.414, not equality.

Now you try: Classify π and 3.14. Are they equal?

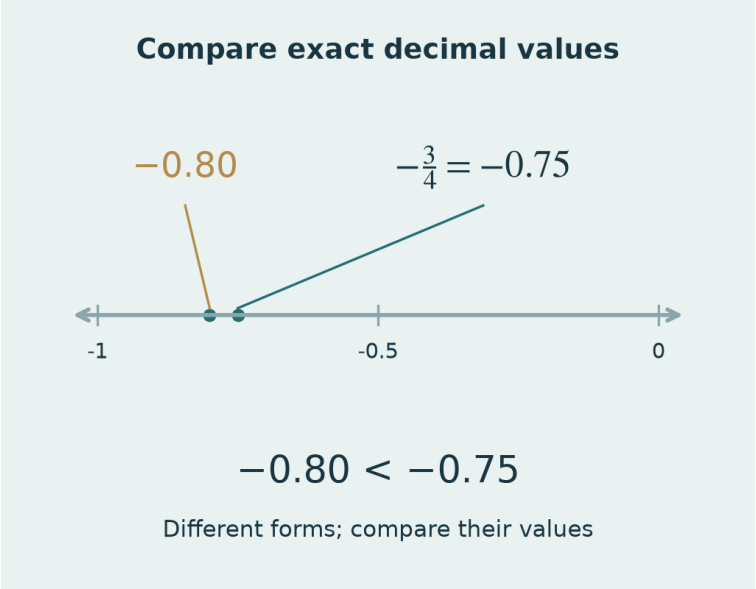
Check after attempting: π is irrational. 3.14=157/50 is rational. They are not equal; π≈3.14 is an approximation.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Compare two negative values

$-\frac{3}{4} = -0.75 < -0.6$



1. Convert the fraction exactly.
2. The smaller value lies farther left.
3. Use < and explain why negative 0.75 is smaller.

Now you try: Which is smaller, -0.8 or -3/4? Show an exact comparison.

Check after attempting: -3/4=-0.75, and -0.80<-0.75. So -0.8 is smaller.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Order a mixed list

$-1.35 < -\frac{5}{4} < -1.2$



1. -5/4=-1.25 and -1.2=-1.20.

2. Order the exact decimal values from left to right.

3. Return the fraction to its original form in the answer.

Now you try: Order -2/3, -0.7 and -3/5 from least to greatest without rounding them to the same decimal.

Check after attempting: Use denominator 30: -0.7=-21/30, -2/3=-20/30 and -3/5=-18/30. Therefore -0.7<-2/3<-3/5.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Read absolute-value boundaries carefully

$$|-8|-|3-9|=8-6=2$$

Inside → absolute value → outside

$$|-8|=8$$

$$3-9=-6$$

$$|3-9|=|-6|=6$$

$$8-6=2$$

Keep the subtraction outside the bars

1. Evaluate inside each pair of bars.
2. Take the non-negative distances 8 and 6.
3. Keep the subtraction outside: 8-6=2.

Now you try: Evaluate |-5+2| and -|-5|. Explain why their answers have different signs.

Check after attempting: |-5+2|=|-3|=3. The other expression is -(5)=-5 because its minus sign is outside the bars.

Worked method & transfer

LESSON 1.1 / Real numbers and the number line

Signed change and direct distance

$-3-5=-8, \qquad |-3-5|=8$



1. Label the initial floor 5 and final floor -3.
2. Final minus initial gives signed change -8 floors.
3. The direct distance is 8 equal floor intervals.

Now you try: A temperature changes from -6°C to 2°C. Find its signed change and the distance between the two readings on the Celsius scale.

Check after attempting: Signed change: 2-(-6)=+8°C. Distance between readings: |2-(-6)|=8°C. The positive sign indicates an increase.

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

01 / Number sets

Give the most specific set for -12. Then name its larger sets.

02 / Number sets

Classify 0 using the convention in this course.

03 / Number sets

Classify $\sqrt{36}$ in every applicable set.

04 / Number sets

Write 0.375 as a simplified fraction and classify it.

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

05 / Rational or irrational

Is $0.\overline{3}$ rational even though it never ends?

06 / Rational or irrational

Classify $\sqrt{2}$ and the exact decimal 1.414.

07 / Rational or irrational

Is $\pi = 3.14$ a correct statement? Repair it.

08 / Rational or irrational

A decimal has a 1 followed by one zero, then a 1 followed by two zeros, then a 1 followed by three zeros, and so on forever: 0.101001000100001…. Is it rational?

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

09 / Order

Which is greater: -7 or -2? Explain using position.

10 / Order

Which is smaller: -0.8 or $-\frac{3}{4}$?

11 / Order

Order $-\frac{5}{4},\ -1.2,\ -1.35$ from least to greatest.

12 / Order

Order $-\frac{2}{3},\ -0.7,\ -\frac{3}{5}$ from least to greatest.

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

13 / Number-line reasoning

Place $-1.5, 0, \frac{1}{2}, 0.5$ on a scaled number line. How many distinct points are needed?

14 / Opposites

Give the opposite and the absolute value of -4.5.

15 / Absolute value

Evaluate $|2 - 11|$.

16 / Absolute value

Evaluate $|-8| - |3 - 9|$.

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

17 / Absolute value

Evaluate $-|-8|$.

18 / Positions

Find every real coordinate x with $|x| = 4$.

19 / Positions

Find every coordinate with $|x| = 0$. Can any coordinate have $|x| = -3$?

20 / Distance

Find the distance between -6 and -2.

Build independence

LESSON 1.1 / Real numbers and the number line

Name: _____ Date: _____

Attempt first. Show your method. Use a notebook for extended explanations.

21 / Change and distance

A lift moves directly from floor 5 to floor -3 with equally spaced floors. Find signed change and distance in floor intervals.

22 / Change and distance

A temperature changes from -6°C to 2°C. Find its signed change. What changes if the readings occur in reverse order?

23 / Challenge

Can two irrational numbers have a rational sum? Give a counterexample to “always irrational”.

24 / Challenge

A point is 4 units from coordinate -1. Find all possible coordinates and check both distances.

Sketch, label & justify

LESSON 1.1 / Real numbers and the number line

Use these spaces for scaled number lines, workings and explanations.

Diagram / givens / conclusion / reason

Diagram / givens / conclusion / reason

Sketch, label & justify

LESSON 1.1 / Real numbers and the number line

Use these spaces for scaled number lines, workings and explanations.

Diagram / givens / conclusion / reason

Diagram / givens / conclusion / reason

Homework · reason clearly

LESSON 1.1 / Real numbers and the number line

Eight problems. Attempt without notes first; then check and record one correction.

QUESTION 1

Classify $-18, 0, \frac{7}{2}, \sqrt{49}$ by their most specific sets, then list all larger applicable sets.

QUESTION 2

Explain why $0.\overline{6}$ is rational. Classify $\sqrt{5}$ and the exact decimal 2.236, and use the correct comparison symbol between them.

Homework · reason clearly

LESSON 1.1 / Real numbers and the number line

Eight problems. Attempt without notes first; then check and record one correction.

QUESTION 3

Order $-\frac{7}{8},\ -0.9,\ -\frac{4}{5}, 0$ from least to greatest. Show an exact comparison and a scaled sketch.

QUESTION 4

Evaluate $|-12|$, $|5-13|$, $|-6|-|2-9|$ and $-|-6|$. Explain the role of each pair of bars.

Homework · reason clearly

LESSON 1.1 / Real numbers and the number line

Eight problems. Attempt without notes first; then check and record one correction.

QUESTION 5

A submarine moves directly from height -25 m to height -7 m relative to sea level. Find signed change and endpoint distance. Then reverse the trip.

QUESTION 6

Find every position 6 units from zero. Find every position 3 units from coordinate -2. Check each answer.

Homework · reason clearly

LESSON 1.1 / Real numbers and the number line

Eight problems. Attempt without notes first; then check and record one correction.

QUESTION 7

Correct these claims with reasons: every square root is irrational; every nonterminating decimal is irrational; absolute value is always positive.

QUESTION 8

Create a rational sum from two irrational numbers, then give an irrational sum from two irrational numbers. Explain why one example cannot establish an “always” rule.